Ship stability and parametric rolling *

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SUMMARY: *Ships, and therefore ship stability, is of vital importance for the transportation of humans and livestock, as well as providing the only means of transporting heavy cargoes between the continents. We present a simple model for ship stability based on the well known Mathieu equation, a second-order differential equation with periodic coefficients, which describes the phenomenon of parametric rolling. Using MATLAB, this model can be used in a classroom setting to introduce students to an important class of differential equations that are not ordinarily taught in the undergraduate engineering curriculum.*

1 INTRODUCTION

The stability of ships is of vital importance to the maritime world, and for a variety of reasons is likely to become even more important in the future. For example, ships in general are becoming ever larger, and a growing number of cruise ships have more decks above the waterline, potentially raising their centre of gravity and therefore decreasing their stability. There is also the growing economic pressure to "sail closer to the wind", and therefore potentially jeopardise stability and hence safety.

A floating ship has six degrees of freedom. In order to completely define the ship's motion it is necessary to consider movements in all these modes. In addition, allowance for specific hull structure and a host of non-linear effects must also be made. Many models, of varying sophistication, have been produced to predict the stability of a ship under a variety of conditions and give very good results when compared to the observed behaviour.

While sophisticated models produce many useful results, in general the problem may only be solved numerically (Silva et al, 2005). As such, the physical insight that these models provide, and therefore their pedagogical use, is limited. On the other hand, one can use a simple model that provides both physical insight and also be useful as a teaching tool.

In this paper we describe the basic concepts of the forces influencing ship stability and introduce the differential equation, Mathieu's equation, describing the motion of a ship from the point of view of stability. We then give some typical numerical solutions of the equation, obtained using MATLAB, and investigate the conditions under which the stability may be aided or compromised.

2 BASIC THEORY

2.1 The forces

If we neglect forces such as wind and wave effects, which are clearly of great importance in all but the best weather conditions, there are two basic forces in action on a ship. These are: (i) the weight force, $W = mg$, acting vertically downwards, where m is the mass of the ship and g is the acceleration due to gravity; and (ii) the buoyancy force, F_B, acting vertically upwards (Paffett, 1990) (see figure 1). The weight force acts through the centre of gravity of the ship, the centre of gravity being coincident with the centre of mass in a uniform gravitational field. The buoyancy force (sometimes called the up-thrust) which, by Archimedes' principle, is equal to the weight of fluid displaced, acts through the centre of gravity of the displaced fluid.

The ship's centre of gravity is essentially fixed (unless the mass distribution in the ship is altered), but the centre of buoyancy may move considerably if, for example, the ship heels (rolls) over. In general, the centre of gravity of the ship and the centre of gravity of the displaced fluid are not located at the same point. Furthermore, except when the ship is in the upright equilibrium position, they usually do not even lie in the same vertical line. If they are not in the same vertical line, then a moment or couple occurs, which may tend either to right the ship returning it

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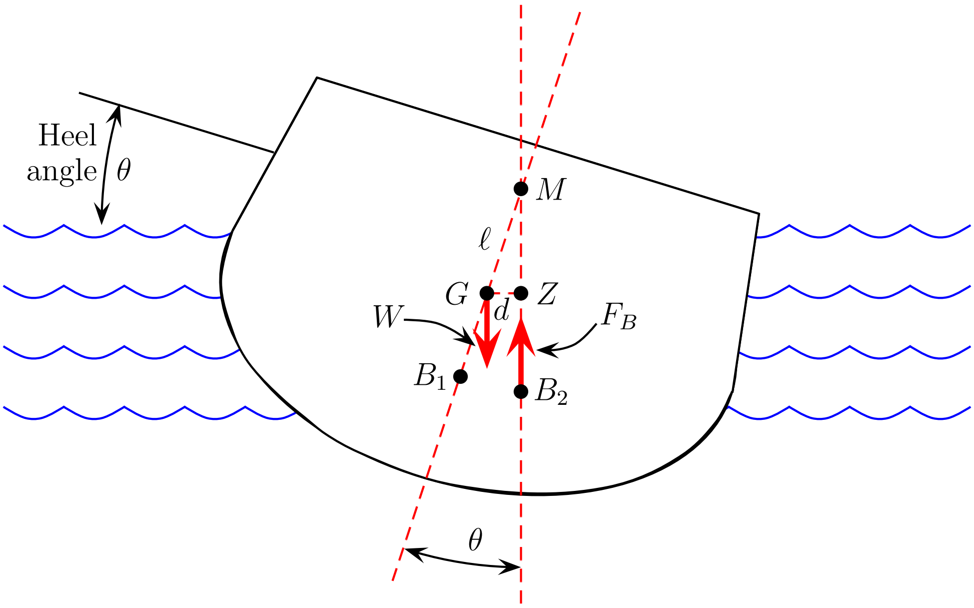

Figure 1: The geometry of a ship shown heeling through an angle θ in the clockwise direction. In this diagram, G denotes the centre of gravity, W is the weight force, F_B is the buoyancy force, and M is the metacentre (see text).

to its equilibrium position or cause it to heel over further, potentially with catastrophic results.

Figure 1 shows a schematic diagram of the basic geometry of a heeling ship and introduces a number of relevant terms.

As a ship heels over, the original centre of buoyancy when the ship is in the upright equilibrium position, B_1, shifts laterally. The point at which a vertical line through the new centre of buoyancy, B_2, intersects a line through the original centre of buoyancy and the ship's centre of gravity, G, is known as the metacentre, M (Paffett, 1990). The metacentre, like the centre of buoyancy, may move as the ship heels. The distance $\overline{GM}$ between the centre of gravity and the metacentre is known as the metacentric height, denoted by l in figure 1. The weight force, $W = mg$, acts vertically downwards through the centre of gravity and the buoyancy force, F_B, acts vertically upwards through the centre of buoyancy. The perpendicular (horizontal) distance $\overline{GZ}$ between the directions of the weight and buoyancy forces is known as the righting arm, and is denoted by d in figure 1.

The ship is shown heeling in a clockwise direction in figure 1. For the situation depicted, the moment produced by the weight and buoyancy forces is in an anticlockwise direction and thus produces a righting effect or restoring moment. If, however, for any reason the centre of buoyancy were to move to the left of the vertical line through the centre of gravity causing M to fall below G, the introduced moment would be in a clockwise direction, thereby producing a potentially unstable situation.

One way of promoting stability is to increase the restoring moment by increasing the length of the righting arm (ie. the distance $\overline{GZ}$). This may be accomplished by having the metacentre well above the centre of gravity, ie. by increasing the length $\overline{GM}$. However, a compromise must be reached because increasing $\overline{GM}$ gives rise to a shorter roll period or "stiffness" of the vessel, which may be disquieting for passengers in the case of a passenger ship. A smaller $\overline{GM}$ results in a longer roll period and a so-called "tender" ship more suitable for passenger comfort, but at the expense of increasing the risk of instability and possible capsize (Von Dokkum, 2008).

2.2 The equation of motion

In the analysis below and also in the numerical phase plane analysis results presented in section 3, we consider two cases, that of natural rolling and of parametric rolling of the ship. In natural rolling, the position of the centre of buoyancy and the metacentre are assumed fixed and hence the length $\overline{GM}$ also remains fixed. In so-called parametric rolling, $\overline{GM}$ may vary, specifically as a result of a wave travelling along the length of the ship, as shown in figure 2, and this may occur even when the wave approaches from dead ahead or dead astern (Paffett, 1990).

2.2.1 Natural rolling

If the location of G and M are known, from figure 1 the righting moment for small θ is given by:

$$M = -Wd = -Wl\sin\theta \approx -Wl\theta \quad (1)$$

where d is the distance $\overline{GZ}$ and l is the distance $\overline{GM}$.

Assume that the ship is floating freely in calm water when it is suddenly disturbed causing it to heel. The motion following the removal of the

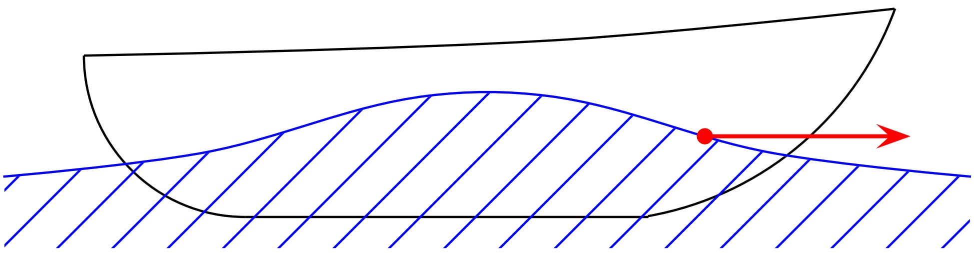

Figure 2: A wave travelling down the length of a ship can give rise to a shift in the metacentre, a consequent change in the distance $\overline{GM}$ (see figure 1), and parametric rolling of the ship from side to side.

disturbing moment follows directly from Newton's second law applied to angular motion and is given by (Molland, 2008):

$$I\frac{d^2\theta}{dt^2}+D\frac{d\theta}{dt}-M=0 \quad (2)$$

Here I is the moment of inertia of the ship; $Dd\theta/dt$ is a (resistive) viscous damping term due to the water, proportional to the angular roll velocity; and the negative sign in the righting moment term arises because of the sign convention used in equation (1). We now define the natural roll frequency, ω_0, by:

$$\omega_0^2=\frac{Wl}{I} \quad (3)$$

which, on substituting for M from equation (1), enables us to write equation (2) as:

$$\frac{d^2\theta}{dt^2}+\mu\frac{d\theta}{dt}+\omega_0^2\theta=0 \quad (4)$$

where $\mu = D/I$. This is the equation of the classic damped harmonic oscillator, which is a staple for any course in differential equations.

2.2.2 *Parametric rolling*

Allievi & Soudack (1990) considered the effect of a two-dimensional wave of period T travelling down the length of a ship (see figure 2). They argued that such a wave will periodically change the distance $\overline{GM}$ in a sinusoidal manner with an amplitude of $\Delta l/2$, where Δl is proportional to the wave height. The righting moment can then be written as:

$$M(t)\approx -W\left(l-\frac{\Delta l}{2}\cos 2\omega t\right)\theta \quad (5)$$

where ω is the angular frequency of the wave. Using equation (5), we write equation (2) as:

$$\frac{d^2\theta}{dt^2}+\left(\frac{D}{I}\right)\frac{d\theta}{dt}+\left(\frac{W}{I}\right)\left[l-\left(\frac{\Delta l}{2}\right)\cos 2\omega t\right]\theta=0 \quad (6)$$

In order to cast this equation into its more usual form, we now make the following transformations: $y = \theta$, $z = \omega t$, $\gamma = (D/\omega I)$, $a = \omega_0^2/\omega^2$ and $q = (b/2)a$, where $b = (\Delta l/2)/l$.

The above differential equation (6) then becomes:

$$\frac{d^2y}{dz^2}+\gamma\frac{dy}{dz}+(a-2q\cos 2z)y=0 \quad (7)$$

Letting $\gamma = 0$ (ie. $D = 0$), leads to the so-called Mathieu equation in the canonical form (McLachlan, 1964):

$$\frac{d^2y}{dz^2}+(a-2q\cos 2z)y=0 \quad (8)$$

The parameter a is the square of the ratio of the natural frequency to the longitudinal wave frequency, and b (thus q) is a measure of the wave's amplitude. Here we take γ, a and q as (dimensionless) parameters to be varied. Note that with $q = 0$, equation (7) reduces to the natural roll case, equation (4), as it must, since $q = 0$ implies that $b = 0$ and hence $\Delta l = 0$.

The Mathieu equation is a particular case of a second-order differential equation with periodic coefficients. What is of interest is to understand the behaviour of the solution of this differential equation according to the values of a and q. Specifically, what condition (ie. what values of a and q) must prevail such that the wave augments the heal angle θ (ie. y) that could lead to over instability and hence capsize. Here, following Allievi & Soudack (1990), we define physical capsize to occur if $\theta > 1$ radian ($\simeq 57°$).

3 PHASE PLANE ANALYSIS

Solutions to Mathieu's equation have been extensively studied and their properties were summarised by Abramowitz & Stegun (1965) and discussed in greater detail in the classic text by McLachlan (1964). Although the general theory based on Floquet theory is beyond the capabilities of most undergraduate students, the use of MATLAB or any other mathematical software package can make the analysis of the Mathieu equation accessible to all students. By plotting trajectories (solutions) in the phase-plane allows students to "see" solutions of the differential equation.

We employ MATLAB to present selected results for natural rolling, for which $b = 0$ (and hence $q = 0$), and parametric rolling, for which this constraint is relaxed.

3.1 Natural rolling

Let us assume that the ship is floating freely in still water when it is suddenly disturbed. From equation (4) and in the absence of damping ($\mu = 0$), we have simple harmonic motion and the ship will oscillate with frequency ω_0. For rolling in a "beam sea", a sea whose surface waves are approximately normal to the course of the ship, equation (4) is augmented by introducing a forcing function on the right-hand side. This leads students nicely to the ideas of beating and the possible catastrophic consequence of resonance if the frequency of the waves is close to ω_0.

The effect of damping is to exponentially modulate the amplitude of oscillations, the ship eventually returning to its upright position. This is due to the transfer of roll energy to the surrounding water by frictional forces. Figure 3 shows the time-series solution for θ (in radian) and the phase-plane for the initial condition $\theta = 0$ and $\theta' = 0.6$.

In the analysis we have taken M to be above G (see figure 1) and this situation is said to be positively stable. When M is below G, the ship is said to be negatively stable and the equation becomes:

$$\frac{d^2\theta}{dt^2} + \mu\frac{d\theta}{dt} - \omega_0^2\theta = 0 \qquad (9)$$

for which the solutions are unbounded hyperbolic functions (for $\mu = 0$), rather than bounded oscillatory functions, and are therefore unstable.

3.2 Parametric rolling

For parametric rolling, we have $b \neq 0$ and hence $q \neq 0$, ie. the metacentre may move due to a longitudinal wave travelling along the length of the ship (see figure 2), and we must employ equation (7).

It is a property of the Mathieu equation that its solution takes different forms according to the values of a and q. The a-q plane, for $q > 0$, is divided into regions in which the solution corresponding to a point (a, q) is either stable or unstable (see figure 4). Figure 5 shows an example of undamped parametric rolling in the case $a = 1.3$ and $q = 0.26$, placing it in a stable region of figure 4. The ship oscillates indefinitely, with the details of the oscillation being strongly dependent on the values of a and q. For this particular case, large amplitude beats occur between the natural roll frequency ω_0 and the disturbing water-wave frequency ω. Typically, the solutions are aperiodic.

If instead we consider the case of $a = 1$ and $q = 0.05$, corresponding to a wave of modest size ($b = 0.1$, representing a 10% change in l) but in an unstable region of figure 4, the amplitude of the oscillations

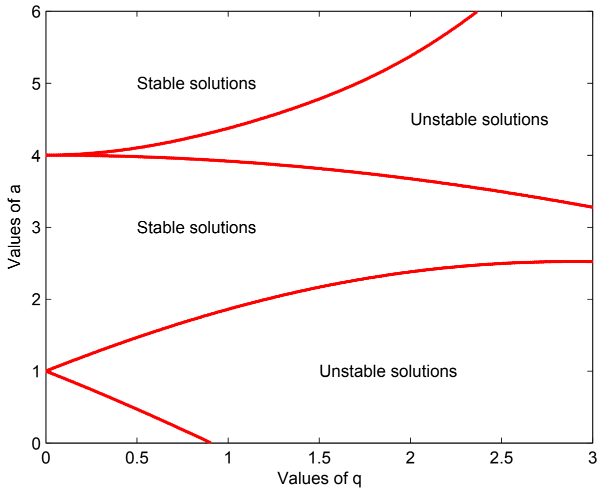

Figure 4: Stability chart for solutions of the Mathieu equation up to the second order in which $a = (\omega_0/\omega)^2$ is plotted as a function of $q = (\Delta l/4l)(\omega_0/\omega)^2$. Note the regions of instability centred on $a = n^2$, ie. $\omega_0 = n\omega$, where $n = 1, 2, 3, \ldots$; the instability increasing as one moves further into the unstable solution region.

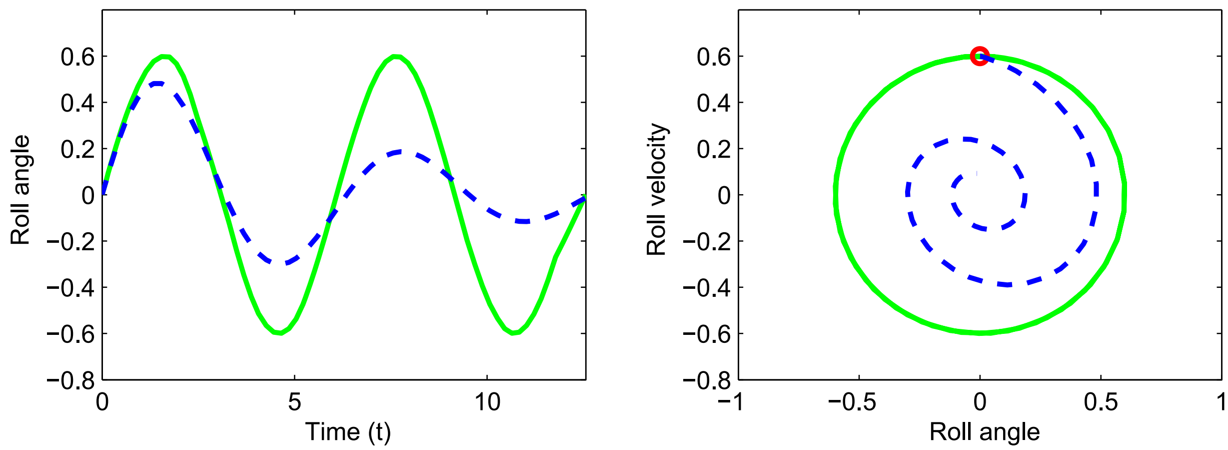

Figure 3: An example of solutions to equation (4) for the natural roll case without damping ($\mu = 0$, solid line) and with damping ($\mu = 0.3$, dashed line), with $\omega_0 = 1$. The initial condition ($\theta = 0$, $\theta' = 0.6$) is marked with an open red circle.

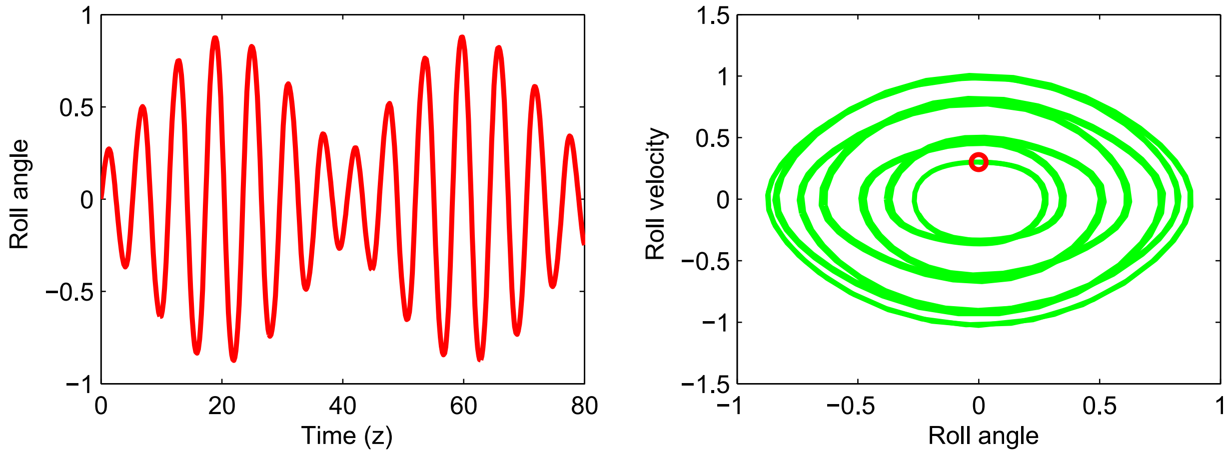

Figure 5: An example of parametric rolling without damping ($\gamma = 0$), as derived from equation (8). Here $a = 1.3$ and $q = 0.26$ with initial conditions of $y = 0$ and $y' = 0.3$.

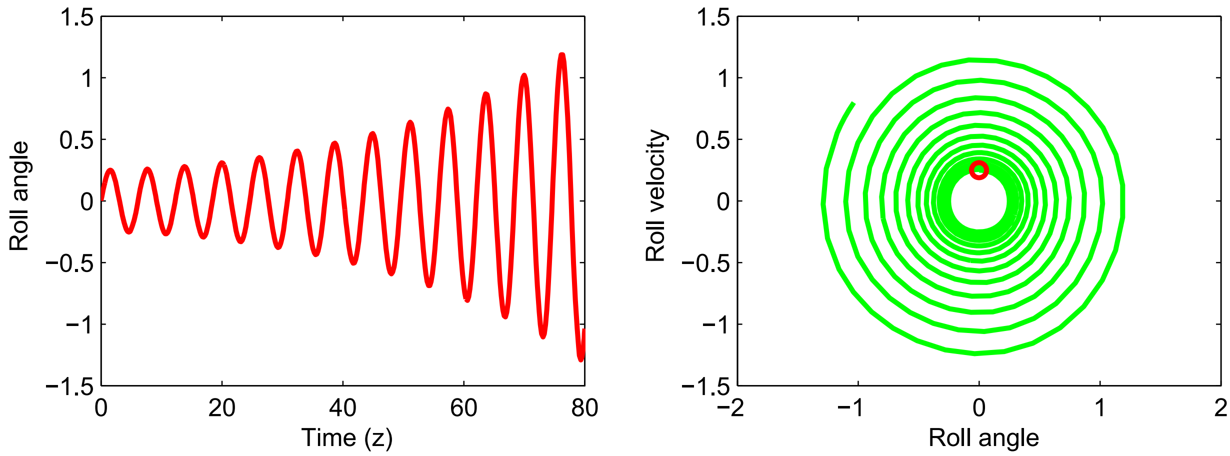

Figure 6: An example of parametric resonance without damping ($\gamma = 0$), as derived from equation (8). Here $a = 1$ and $q = 0.05$, with initial conditions of $y = 0$ and $y' = 0.25$.

increases in a manner reminiscent of resonance (see figure 6). One would hope that students will notice this to be quite strange as Mathieu's equation (8) is homogeneous and therefore this behaviour must be attributed to something different from regular resonance. This so-called parametric resonance is due to the time varying nature of the coefficients. Indeed, the observation of beating in figure 5 is also a manifestation of the periodic coefficients appearing in the differential equation.

As figure 4 reveals, the Mathieu equation exhibits instabilities when the natural roll frequency, ω_0, is close to an integral multiple of the driving waterwave frequency, ω, ie. when $a = (\omega_0/\omega)^2 \approx n^2$, where $n = 1, 2, 3, \ldots$ (see Figure 8A of McLachlan (1964) for further details). Of these resonances, the first ($n = 1$) and second ($n = 2$) are the most critical for a ship, since in practice $\omega_0 \lesssim 3\omega$. For these values of ω, instabilities can be generated with waves of modest amplitude (Paffett, 1990; Obreja et al, 2008). The result is that resonance in roll is a common phenomenon at sea. According to figure 6, the ship's amplitude of oscillation increases with every cycle leading to capsizing ($y > 1$) at $z \approx 60$. Ships change course to avoid the effect of parametric resonance and thus avoid the embarrassment of capsizing.

The effect of damping is to squeeze the instability regions making them narrower (Allievi & Soudack, 1990; McLachlan, 1964). However, there always exists a threshold value $b_t(n)$ of the wave amplitude, and hence q, necessary for the occurrence of the n^{th}-order parametric resonance, but this value grows rapidly for increasing n. In the case $a = 1$ and $q = 0.05$, a small damping coefficient, $\gamma = 0.025$, is sufficient to stabilise the ship (see figure 7). For $q > 0.05$, parametric resonance results again. In figure 7, the long-term behaviour is not a limit cycle, as the equation we are dealing with is linear. Modelling over a much longer time interval shows the amplitude is decreasing, indeed $y \to 0$ as $z \to \infty$.

4 IN THE CLASSROOM

For several years we have offered projects to our first-year students during the last three weeks of their mathematics course. The main motivation is to convey the central importance of mathematics in our increasingly complex and technological world.

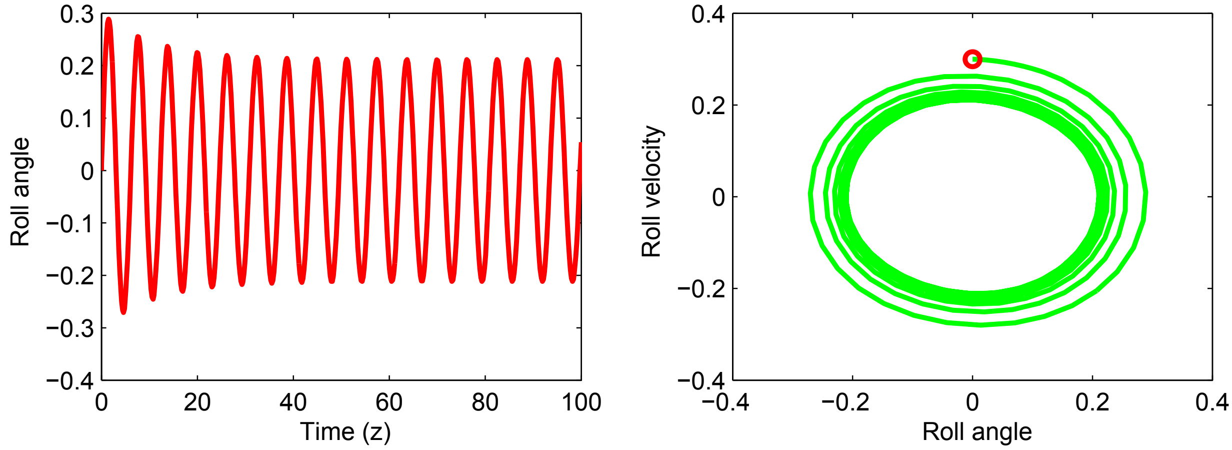

Figure 7: As for figure 6, but derived from equation (7) with a damping factor of $\gamma = 0.025$. The amplitude of the oscillations is decreasing, although very slowly.

The project *Ship Stability* has always attracted strong interest from students. In the project, students are provided with a simple MATLAB code to solve a second-order differential equation with constant coefficients using the MATLAB in-built solver ODE45. They are required to modify the program to solve the Mathieu equation and reproduce selected results of Allievi & Soudack (1990). As part of their report, they also need to demonstrate their understanding of the code by describing what it does[1]. With a little assistance, based on the material presented in this paper, students manage to generate similar (but sometimes not accurate) plots to those given by Allievi & Soudack (1990). In the paper by Allievi & Soudack (1990), the initial conditions are not always explicitly given, hence students often use physically incorrect conditions with results indicating their ship has revolved several times about its centre of gravity.

Mathieu's equation is a particular equation of the more general form:

$$\frac{d^2y}{dz^2} + p(z)y = 0$$

where $p(z)$ is a periodic function in z. One of the advantages of a graphical approach is that students can experiment with the parameters of a differential equation, learn to make good conjectures, design experiments to test their conjectures, and interpret the results of their experiments, which do not require strong algebraic skills. Questions we wish to explore and have partially addressed in this paper are: What happens when $p(z) > 0$ for $z > 0$? Are the solutions oscillatory? Are they periodic? Would allowing $p(z) = 0$ at a finite set of points affect the qualitative behaviour of the solution? Are oscillatory solutions possible if we allow $p(z) < 0$ for some intervals? These ideas are briefly discussed in a nice (short) article by Park (1997). He wrote: "The importance of the technology is not that it can give answers to these questions, but that it allows us to ask these questions in the first place", with which we agree.

[1] Alternatively, John Polking from Rice University has written a MATLAB code, ODESOLVE, that solves non-autonomous systems. It is window driven and simple to use. It can be downloaded from www.math.rice.edu/~dfield.

5 CONCLUDING REMARKS

The aim of this paper is not to develop an in-depth analysis of ship stability, as this requires an understanding of the analytical tools for solving the Mathieu equation. Rather the use of the model of ship stability presented here is intended to motivate the inclusion of second-order differential equations with periodic coefficients into the curriculum. McLachlan (1964) provided several other applications, with more contemporary applications provided by Ruby (1996).

We note that differential equations with variable coefficients are not totally alien to our engineering students. For instance, students encounter Bessel's equation in modelling the vibration of objects that are spatially confined, eg. a drum. However, the Mathieu equation represents a special class of differential equations that warrants further attention.

It has been our experience that even students that exhibit low algebraic skill are nevertheless capable of learning relatively complex concepts. This suggests to us that teaching concepts for understanding does not always have to wait until all relevant procedural skills are mastered. However, we do recommend that differential equations such as the Mathieu equation be taught after the study of equations with constant coefficients. This will provide the necessary foundations for an understanding of the types of solution behaviour discussed in this paper.

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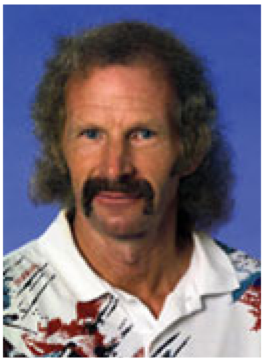

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